Probabilistic Methods in Combinatorics

Solutions to Assignment 11

Problem 1. For a graph H, let $\alpha(H)$ be its independence number. Show that for the random graph $G \sim G(n,p)$, we have

$$\mathbb{P}(|\alpha(G) - \mathbb{E}[\alpha(G)]| \geq \sqrt{n \log n}) = o(1).$$

Solution. Note that $\alpha(G)$ is 1-Lipschitz with respect to vertex exposure. Therefore, by Azuma-Hoeffding, we have

$$\mathbb{P}(|\alpha(G) - \mathbb{E}[\alpha(G)]| \geq \sqrt{n \log n}) \leq 2\exp(-\frac{n \log n}{2n}) = o(1),$$

as wanted.

Problem 2. Prove that there is an absolute constant c such that for every $n > 1$ there is an interval I_n of at most $c\sqrt{n}/\log n$ consecutive integers such that the probability that the chromatic number of $G(n, 1/2)$ lies in I_n is at least 0.99.

Solution. Let u be the least integer such that

$$\mathbb{P}[\chi(G) \leq u] \geq \frac{1}{300}. \tag{1}$$

Let Y be the size a smallest set of vertices S such that $\chi(G \setminus S) \leq u$.

Modifying the edges of G at a fixed vertex changes Y by at most 1, so Y is 1-Lipschitz with respect to vertex exposure. Let λ be such that $2e^{-\lambda^2/2} = 1/300$. Then by the Azuma-Hoeffding inequality,

$$\mathbb{P}\left[|Y - \mathbb{E}[Y]| > \lambda\sqrt{n}\right] < 2e^{\frac{-\lambda^2}{2}} = \frac{1}{300}. \tag{2}$$

I claim that $\mathbb{E}[Y] \leq \lambda\sqrt{n}$. Indeed, otherwise,

$$\mathbb{P}[Y = 0] \leq \mathbb{P}\left[\mathbb{E}[Y] - Y \geq \mathbb{E}[Y]\right] \leq \mathbb{P}\left[\mathbb{E}[Y] - Y > \lambda\sqrt{n}\right] < \frac{1}{300}.$$

But the event $\{Y = 0\}$ is exactly the event $\{\chi(G) \leq u\}$, which has probability at least $1/300$, by the choice of u. Thus $\mathbb{E}[Y] \leq \lambda\sqrt{n}$, as claimed.

It follows from the inequality (2) that

$$\mathbb{P}\left[Y \geq 2\lambda\sqrt{n}\right] \leq \mathbb{P}\left[Y - \mathbb{E}[Y] \geq \lambda\sqrt{n}\right] < \frac{1}{300}. \tag{3}$$

To finish, I claim that there exists a constant $c > 0$ such that

$$\mathbb{P}\left[\text{there is a set } S \text{ of size at most } 2\lambda\sqrt{n} \text{ with } \chi(G[S]) \geq \frac{c\sqrt{n}}{\log n}\right] \leq \frac{1}{300}. \tag{4}$$

Before proving (4), let us see how to use it to finish the proof. By (1), (3) and (4), we see that with probability at least $1 - 3/300 = 0.99$ the following three properties hold.

- $\chi(G) \geq u$,
- there is a set S of size at most $2\lambda\sqrt{n}$ such that $\chi(G \setminus S) \leq u$,
- $\chi(G[S]) \leq \frac{c\sqrt{n}}{\log n}$.

In particular,

$$u \leq \chi(G) \leq \chi(G \setminus S) + \chi(G[S]) \leq u + \frac{c\sqrt{n}}{\log n}.$$

So we know that with probability at least 0.99 the chromatic number of $G(n, 1/2)$ is in the interval $[u, u + c\sqrt{n}/\log n]$.

Now, let us turn to the proof of (4). Recall Theorem 7.7 from lectures notes:

$$\mathbb{P}\left[\alpha(G(m, 1/2)) \leq k\right] \leq e^{-m^{2+o(1)}},$$

where $k = (1 + o(1))2\log_2 m$. Using the fact that $k \geq \log m$, we have

$$\mathbb{P}\left[\alpha(G(m, 1/2)) \leq \log m\right] \leq e^{-m^{2+o(1)}}.$$

By plugging in $m = n^{1/3}$, and using the union bound, it follows that the probability that there is a set of vertices S of size at least $n^{1/3}$ that does not contain an independent set of size at least $\frac{1}{3}\log n$ is at most

$$\binom{n}{n^{1/3}} e^{-n^{2/3+o(1)}} \leq n^{n^{1/3}} e^{-n^{2/3+o(1)}} \leq \exp(n^{1/3}\log n - n^{2/3+o(1)}) = o(1).$$

In particular, with probability at least $1/300$ every set of size at least $n^{1/3}$ contains an independent set of size at least $\frac{1}{3}\log n$. Now, let S be a set of size at most $2\lambda\sqrt{n}$, and write

$S_0 = S$. Repeat the following for $i \geq 1$.

If $|S_i| \leq n^{1/3}$, stop.

Otherwise, let U_i be an independent set of size at least $\dfrac{1}{3} \log n$ in S_i and set $S_{i+1} = S_i \setminus U_i$.

(Note that this is possible because we assumed that every set of size at least $n^{1/3}$ has an independent set of size at least $\frac{1}{3} \log n$.) Let l be the value of $i - 1$ at the end of the procedure. Then $l \leq 6\lambda\sqrt{n}/\log n$, and let $W = S \setminus (U_1 \cup \ldots \cup U_l)$. We obtain a proper colouring of $G[S]$ with at most $6\lambda\sqrt{n}/\log n + n^{1/3}$ by colouring each U_i with a distinct colour and colouring each of the vertices of W with a distinct new colour. It follows that $\chi(G[S]) \leq 6\lambda\sqrt{n}/\log n + n^{1/3} \leq 7\lambda\sqrt{n}/\log n$. Setting $c = 7\lambda$, (4) holds.